April 19

Announcements

- Quiz 2 today

Recap

Let $K \subset L$ be a field extension.

DEF:

1. We say $K \subset L$ is finite if $|L:K|$ is finite
   (equiv, L finite dim'l K-vector space)
2. We say $K \subset L$ is algebraic if every $\alpha \in L$ is algebraic over K
3. We say $K \subset L$ is transcendental if not algebraic

Prop: $K \subset L$ finite field extension $\Rightarrow$ $K \subset L$ algebraic

Proof: Pick $\alpha \in L$

Consider $1, \alpha, \alpha^2, \ldots,$

Once we take $n \geq |L:K|$

$\underbrace{1, \alpha, \alpha^2, \ldots, \alpha^n}_{n+1 \text{ elements}}$ are lin. dependent

Pick k minimal such that $1, \alpha, \ldots, \alpha^k$ are lin. dep.

Therefore, $\exists\, c_0, \ldots, c_k \in K$ not all zero such that

$$c_0 \cdot 1 + c_1 \alpha + \cdots + c_k \alpha^k = 0$$

Then define

$$f(x) = c_0 + c_1 x + \cdots + c_k x^k$$

So $f(\alpha) = 0 \Rightarrow \alpha$ is algebraic

Rmk: Can arrange $c_k = 1$!

Cor: Picking k minimal such that $1, \alpha, \ldots, \alpha^k$ are lin dep via $c_0 + c_1 \alpha + \cdots + c_k \alpha^k$ with $c_k = 1$, the $f(x) = c_0 + c_1 x + \cdots + x^k$ is the min poly of α.

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Pf: To show f is irreducible, suppose $f = f_1 \cdot f_2$ with $\deg f_1, \deg f_2 < k$.

Then α is a root of f_1 or f_2.

Let's assume $f_1(\alpha) = 0$.

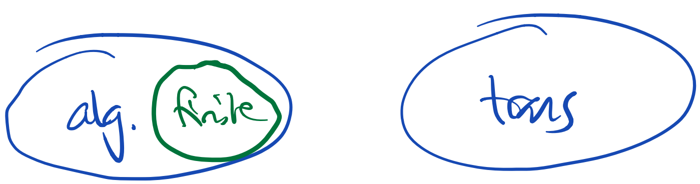
With $f_1 = x^j + d_{j-1}x^{j-1} + \dots + d_0$

Know $j < k$ &

$$f_1(\alpha) = \alpha^j + d_{j-1}\alpha^{j-1} + \dots + d_0 = 0$$

$\Rightarrow 1, \alpha, \dots, \alpha^j$ are lin dep.

contradicting minimality of k



Summary: finite $\Rightarrow$ algebraic

Ex:

① $\mathbb{R} \subset \mathbb{C}$: finite & algebraic

② $\mathbb{Q} \subset \mathbb{C}$: transcendental

$\pi \in \mathbb{C}$

③ $\mathbb{Q} \subset \underbrace{\mathbb{Q}(\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \dots)}$

Is it algebraic? Yes!

Surely, $\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \dots$ are all algebraic

Is ~~it~~ finite? No!

④ $\mathbb{Q} \subset \overline{\mathbb{Q}} = \{\alpha \in \mathbb{C} \text{ algebraic}/\mathbb{Q}\}$

algebraic
not finite

⑤ $\mathbb{C} \subset \mathbb{C}(x) := \left\{ \frac{f(x)}{g(x)} \;\middle|\; f, g \in \mathbb{C}[x],\ g \neq 0 \right\}$

$x \in \mathbb{C}(x)$ transcendental

(can't happen $x^k + c_{k-1}x^{k-1} + \dots + c_0 = 0$, $c_i \in \mathbb{C}$

View "x" as a formal variable

Keep in mind, if $K \subset L$ field ext and $\alpha_1, \dots, \alpha_n \in L$, then

$K(\alpha_1, \dots, \alpha_n) \subset L$ denotes the smallest field ext of K containing $\alpha_1, \dots, \alpha_n$.

Ex: ① $\mathbb{Q} \subset \mathbb{C}$, $\pi \in \mathbb{C}$

$\mathbb{Q}(\pi) \subset \mathbb{C}$ field extension

② x formal variable

$$\mathbb{Q}(x) = \left\{ \frac{f(x)}{g(x)} \;\middle|\; f, g \in \mathbb{Q}[x],\ g \neq 0 \right\}$$

FACT: $\mathbb{Q}(\pi) \cong \mathbb{Q}(x)$

More generally, we have

Ex 3: $K \subset K(x,y) = \left\{ \frac{f(x,y)}{g(x,y)} \;\middle|\; \dots \right\}$

Prop: Let $K \subset L$ field ext. Let $\alpha \in L$ be an element. Then either

(a) $\alpha \in L$ algebraic over K in which case $K(\alpha)$ finite over K

(b) $\alpha \in L$ transcendental over K & $K(\alpha) \cong K(x)$

Proof: For $\alpha \in L$, define

$$K[x] \xrightarrow{\phi} L \quad \text{ring hom}$$

$$f(x) \longmapsto f(\alpha)$$

(image of basis/K $1, x, x^2, \dots$ is $1, \alpha, \alpha^2, \alpha^3, \dots \in L$)

α algebraic $\iff \exists f(x) \neq 0$ $f(\alpha) = 0$

$\iff \ker(\phi) \neq 0$

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α algebraic $\iff \exists f(x) \neq 0$ $f(\alpha) = 0$

$\iff \ker(\phi) \neq 0$

α algebraic $\implies$ ϕ induces an isomorphism

$$\begin{array}{ccc} K[x]/\ker(\phi) & \xrightarrow{\sim} & \operatorname{im}(\phi) \\ \| & & \| \\ K[x]/(f) & \xrightarrow{\sim} & K(\alpha) \end{array}$$

fin. dim'l vect space

where f min poly

If α is not algebraic, then

$\phi : K[x] \hookrightarrow L$ injective

$\implies$ induces an injection

$$K(x) \hookrightarrow L$$

$$\frac{f}{g} \longmapsto \frac{\phi(f)}{\phi(g)}$$

$$x \longmapsto \phi(x) = \alpha$$

$\boxed{\text{B/c } g \neq 0,\ \phi(g) \neq 0 \text{ so makes sense}}$

induces an isom

$$K(x) \xrightarrow{\phi} K(\alpha) \subset L$$

Def: Say $K \subset L$ is a finitely generated field extension if

$\exists\, \alpha_1, \dots, \alpha_n \in L$ such that

$$L = K(\alpha_1, \dots, \alpha_n)$$

Ex: $\mathbb{C} \hookrightarrow \mathbb{C}(x,y)$ fin. gen (but not finite)

Prop: For a field ext $K \subset L$

$K \subset L$ finite $\Longleftrightarrow$ $K \subset L$ algebraic & fin. generated

Ex: $\cdot\ \mathbb{Q} \subset \mathbb{Q}[\sqrt[n]{2} : n \in \mathbb{Z}_{>0}]$ not fin. generated

$\cdot\ \mathbb{Q} \subset \overline{\mathbb{Q}}$ not fin. generated

Pf: Let's skip it.

Prop: Let $K \subset L$ be a field ext.

Then if $\alpha, \beta \in L$ are algebraic over K

so are $\alpha\beta$, $\alpha+\beta$, $\alpha-\beta$, α/β.

Ex: $\sqrt{2}, \sqrt{3}$ alg$/\mathbb{Q}$ $\Rightarrow$ $\sqrt{2}+\sqrt{3}$ alg.

Pf: Consider

$$K \subset K(\alpha) \subset K(\alpha,\beta) \subset L$$

Last time:

$$|K(\alpha,\beta) : K| = \underbrace{|K(\alpha,\beta) : K(\alpha)|}_{\substack{\text{finite b/c} \\ \beta \text{ alg}/K \\ \Rightarrow \beta \text{ alg}/K(\alpha)}} \cdot \underbrace{|K(\alpha) : K|}_{\substack{\text{finite b/c} \\ \alpha \text{ alg}}}$$

So $K(\alpha,\beta)$ finite over K

$$K \subset K(\alpha+\beta) \subset K(\alpha,\beta)$$

$$K(\alpha\beta)$$

$\vdots$

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Then if $\alpha, \beta \in L$ are algebraic over K

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So $K(\alpha,\beta)$ finite over K

$$K \subset K(\alpha+\beta) \subset K(\alpha,\beta)$$

$$|K(\alpha,\beta)|$$

$\vdots$

$\Rightarrow |K(\alpha+\beta):K|$ finite

$\Rightarrow \alpha+\beta$ algebraic

Same for $\alpha-\beta$, $\alpha\beta$, α/β.

Cor: Given $K \subset L$ field ext,

Then the subset

$$F = \{\alpha \in L \mid \alpha \text{ algebraic}/K\} \subset L$$

is a field extension of K.

$$(K \subset F \subset L)$$